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Correction to: L. Demasi, "Introduction to Unsteady Aerodynamics and Dynamic Aeroelasticity", Springer 2024, DOI https://doi.org/10.1007/978-3-031-50054-1

January 10th, 2026

Dear colleague/reader:
First, I would like to thank you for having read the book.

This file contains corrections I implemented in the book since its publication in 2024. I express here my thanks and gratitude to the readers who provided a careful and detailed feedback in the past few months.

To read the corrections the following should be considered.

- The corrections in the text or equations are shown in red on the original book source files. Thus, the pages appear graphically different (including page numbering) than the published book but the content is the same.
- The book contains boxed equations for key relationships. Those expressions have a red description on the right of the box. That should **not** be understood as correction.
- The blue text is **not** a correction. It corresponds to relevant technical points, which are shown in sans serif in the formally published book and in blue text in the original source files.
- Red text in a figure's caption indicates that the figure has been updated

I plan to update this erratum periodically.

I hope that this set of corrections benefits your work. If you identify formulas or text that should be reviewed/corrected, please contact me at ldemasi@sdsu.edu.

Sincerely,

Luciano Demasi

Corrections to

Chapter 19

where $\boldsymbol{w}^{\text{ss dl}}_{\text{nw pg}}$ is the dimensionless normalwash in the x, y, z frame. Each entry is given by the following relationship:

$$w^{\text{ss dl}}_{\text{nw pg}I} = \frac{1}{V_\infty} \left.\frac{\partial \phi^{\text{com}}}{\partial z}\right|_{P^I_C} = \left.\frac{\mathrm{d}c^I_{\text{pg}}}{\mathrm{d}x}\right|_{P^I_C} \tag{19.82}$$

c^I_{pg} is the curve intersection between the actual wing surface and a plane perpendicular to the panel and passing through load and control points. Notice that this is in the original frame. It is important to observe that the matrix of influence coefficients $\boldsymbol{A}^{\text{vl}}_{\text{pg}}$ valid for the *compressible case* is not known.

The transformation of coordinates shown in equation 19.78 is performed and the vortex lattice equation 19.51 is written in the X, Y, Z frame:

$$\boxed{\boldsymbol{A}^{\text{vl}} \cdot \Delta \underline{\boldsymbol{P}}^{\text{ss dl}} = \boldsymbol{w}^{\text{ss dl}}_{\text{nw}}} \tag{19.83}$$

where

$$w^{\text{ss dl}}_{\text{nw}I} := \frac{1}{V_\infty} \left.\frac{\partial \phi^{\text{inc}}}{\partial Z}\right|_{P^I_C} \tag{19.84}$$

Notice that having $X = x$ (see the transformation 19.78) does not imply equality of slopes:

$$\left.\frac{\mathrm{d}c^I}{\mathrm{d}X}\right|_{P^I_C} \neq \left.\frac{\mathrm{d}c^I_{\text{pg}}}{\mathrm{d}x}\right|_{P^I_C} \tag{19.85}$$

In principle the dimensionless normalwashes corresponding to the different coordinate systems are different:

$$w^{\text{ss dl}}_{\text{nw}I} \neq w^{\text{ss dl}}_{\text{nw pg}I} \tag{19.86}$$

There is a relationship between the normalwashes expressed in the 2 coordinates systems. Thus, it can be inferred that

$$\boldsymbol{w}^{\text{ss dl}}_{\text{nw}} = f\left(\boldsymbol{w}^{\text{ss dl}}_{\text{nw pg}}\right) \tag{19.87}$$

At the control point of the receiving panel I, the dihedral is known and equal to γ_I (see Figure 19.6). Thus, the normal and tangent to the reference surface σ_{ref} at the control point can be written as shown below:

$$\begin{aligned} \boldsymbol{n}^I &= -\sin\gamma_I \boldsymbol{e}_2 + \cos\gamma_I \boldsymbol{e}_3 \\ \boldsymbol{\tau}^I &= \cos\gamma_I \boldsymbol{e}_2 + \sin\gamma_I \boldsymbol{e}_3 \end{aligned} \tag{19.88}$$

Given the coordinates x_I, y_I, and z_I of the receiving point, it is convenient to define a local coordinate system x', y', and z' parallel to the global coordinate system but centered at the receiving point P^I_C:

$$x' = x - x_I \qquad y' = y - y_I \qquad z' = z - z_I \tag{19.89}$$

An additional coordinate system is now defined. It is indicated with x'', y'', and z''. This coordinate system is obtained by a rotation of x', y', and z' by and γ_I with respect to x' axis. Thus, it is (see equation 19.88)

$$\begin{aligned} x'' &= x' \\ z'' &= -y' \sin\gamma_I + z' \cos\gamma_I \\ y'' &= y' \cos\gamma_I + z' \sin\gamma_I \end{aligned} \tag{19.90}$$

or, using equation 19.89:

$$\begin{aligned} x'' &= x - x_I \\ z'' &= -\left(y - y_I\right)\sin\gamma_I + \left(z - z_I\right)\cos\gamma_I \\ y'' &= +\left(y - y_I\right)\cos\gamma_I + \left(z - z_I\right)\sin\gamma_I \end{aligned} \tag{19.91}$$

or

$$T = \sqrt{\left(\frac{\partial \sigma}{\partial y}\sin\gamma_I - \frac{\partial \sigma}{\partial z}\cos\gamma_I\right)^2 + \left(\frac{\partial \sigma}{\partial x}\right)^2} \tag{19.97}$$

We then have the unit vector written as follows:

$$\boldsymbol{e}_T := \frac{\boldsymbol{T}}{T} = \frac{\left(\frac{\partial \sigma}{\partial y}\sin\gamma_I - \frac{\partial \sigma}{\partial z}\cos\gamma_I\right)}{T}\boldsymbol{e}_1 - \frac{\frac{\partial \sigma}{\partial x}\sin\gamma_I}{T}\boldsymbol{e}_2 + \frac{\frac{\partial \sigma}{\partial x}\cos\gamma_I}{T}\boldsymbol{e}_3 \tag{19.98}$$

Using equation 19.88 the tangent unit vector is written in the plane Ξ_I:

$$\boldsymbol{e}_T = \frac{\left(\frac{\partial \sigma}{\partial y}\sin\gamma_I - \frac{\partial \sigma}{\partial z}\cos\gamma_I\right)}{T}\boldsymbol{e}_1 + \frac{1}{T}\frac{\partial \sigma}{\partial x}\boldsymbol{n}^I \tag{19.99}$$

By definition of unit vector tangent to the curve, we deduce that the derivative of c^I_{pg} is the ratio between the component in the z'' direction and the component in the x direction:

$$\left.\frac{\mathrm{d}c^I_{\mathrm{pg}}}{\mathrm{d}x}\right|_{P^I_C} = \frac{\left.\frac{\partial \sigma}{\partial x}\right|_{P^I_C}}{\left(\left.\frac{\partial \sigma}{\partial y}\right|_{P^I_C}\sin\gamma_I - \left.\frac{\partial \sigma}{\partial z}\right|_{P^I_C}\cos\gamma_I\right)} \tag{19.100}$$

We now need to work on the X, Y, Z frame and perform the very same steps. First, we observe that the normal to the reference surface σ_{ref} is (notice that it does not have any x component by definition):

$$\boldsymbol{n}(y,z) = \frac{\frac{\partial \sigma_{\mathrm{ref}}}{\partial y}\boldsymbol{e}_2 + \frac{\partial \sigma_{\mathrm{ref}}}{\partial z}\boldsymbol{e}_3}{\sqrt{\frac{\partial \sigma_{\mathrm{ref}}}{\partial y}^2 + \frac{\partial \sigma_{\mathrm{ref}}}{\partial z}^2}} \tag{19.101}$$

and by definition, from the normal it is possible to obtain the sine and cosine values of the dihedral:

$$\cos\gamma_I = \frac{\left.\frac{\partial \sigma_{\mathrm{ref}}}{\partial z}\right|_{P^I_C}}{\sqrt{\left.\left(\frac{\partial \sigma_{\mathrm{ref}}}{\partial y}\right)^2\right|_{P^I_C} + \left.\left(\frac{\partial \sigma_{\mathrm{ref}}}{\partial z}\right)^2\right|_{P^I_C}}} \tag{19.102}$$

$$\sin\gamma_I = -\frac{\left.\frac{\partial \sigma_{\mathrm{ref}}}{\partial y}\right|_{P^I_C}}{\sqrt{\left.\left(\frac{\partial \sigma_{\mathrm{ref}}}{\partial y}\right)^2\right|_{P^I_C} + \left.\left(\frac{\partial \sigma_{\mathrm{ref}}}{\partial z}\right)^2\right|_{P^I_C}}} \tag{19.103}$$

In the X, Y, Z plane we have the correction 19.78. Thus, the reference and actual wing surfaces are now a function of X, Y, Z:

$$\sigma_{\mathrm{ref}}(x,y,z) = \sigma_{\mathrm{ref}}\left(X, \frac{Y}{\beta_\infty}, \frac{Z}{\beta_\infty}\right) =: \sigma^\star_{\mathrm{ref}}(X,Y,Z) : \qquad \sigma(x,y,z) = \sigma\left(X, \frac{Y}{\beta_\infty}, \frac{Z}{\beta_\infty}\right) =: \sigma^\star(X,Y,Z) \tag{19.104}$$

Now when the equations are calculated at the control point it is implied that the values of X, Y, Z are the ones corresponding to the point in the original coordinate system.

If the dihedral is indicated with $\gamma^\star_I$, the analogues of equation 19.100 is now the following:

$$\left.\frac{\mathrm{d}c^I}{\mathrm{d}X}\right|_{P^I_C} = \frac{\left.\frac{\partial \sigma^\star}{\partial X}\right|_{P^I_C}}{\left(\left.\frac{\partial \sigma^\star}{\partial Y}\right|_{P^I_C}\sin\gamma^\star_I - \left.\frac{\partial \sigma^\star}{\partial Z}\right|_{P^I_C}\cos\gamma^\star_I\right)} \tag{19.105}$$

and the counterparts of equations 19.102 and 19.103 are presented below:

$$\cos\gamma_I^\star = \frac{\left.\dfrac{\partial\sigma_{\text{ref}}^\star}{\partial Z}\right|_{P_C^I}}{\sqrt{\left.\left(\dfrac{\partial\sigma_{\text{ref}}^\star}{\partial Y}\right)^2\right|_{P_C^I} + \left.\left(\dfrac{\partial\sigma_{\text{ref}}^\star}{\partial Z}\right)^2\right|_{P_C^I}}} \tag{19.106}$$

$$\sin\gamma_I^\star = -\frac{\left.\dfrac{\partial\sigma_{\text{ref}}^\star}{\partial Y}\right|_{P_C^I}}{\sqrt{\left.\left(\dfrac{\partial\sigma_{\text{ref}}^\star}{\partial Y}\right)^2\right|_{P_C^I} + \left.\left(\dfrac{\partial\sigma_{\text{ref}}^\star}{\partial Z}\right)^2\right|_{P_C^I}}} \tag{19.107}$$

The following is now observed:

$$\frac{\partial\sigma_{\text{ref}}^\star}{\partial X} = \frac{\partial\sigma_{\text{ref}}\left(X, \frac{Y}{\beta_\infty}, \frac{Z}{\beta_\infty}\right)}{\partial X} = \frac{\partial\sigma_{\text{ref}}\left(X(x), \frac{Y(y)}{\beta_\infty}, \frac{Z(z)}{\beta_\infty}\right)}{\partial x}\frac{\partial x}{\partial X} = \frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial x}\frac{\partial x}{\partial X} = \frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial x} \tag{19.108}$$

Thus,

$$\boxed{\left.\frac{\partial\sigma_{\text{ref}}^\star}{\partial X}\right|_{P_C^I} = \left.\frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial x}\right|_{P_C^I}} \tag{19.109}$$

Similarly:

$$\frac{\partial\sigma_{\text{ref}}^\star}{\partial Y} = \frac{\partial\sigma_{\text{ref}}\left(X, \frac{Y}{\beta_\infty}, \frac{Z}{\beta_\infty}\right)}{\partial Y} = \frac{\partial\sigma_{\text{ref}}\left(X(x), \frac{Y(y)}{\beta_\infty}, \frac{Z(z)}{\beta_\infty}\right)}{\partial y}\frac{\partial y}{\partial Y} = \frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial y}\frac{\partial y}{\partial Y} = \frac{1}{\beta_\infty}\frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial y} \tag{19.110}$$

which implies

$$\boxed{\left.\frac{\partial\sigma_{\text{ref}}^\star}{\partial Y}\right|_{P_C^I} = \frac{1}{\beta_\infty}\left.\frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial y}\right|_{P_C^I}} \tag{19.111}$$

Operating with the same approach:

$$\frac{\partial\sigma_{\text{ref}}^\star}{\partial Z} = \frac{\partial\sigma_{\text{ref}}\left(X, \frac{Y}{\beta_\infty}, \frac{Z}{\beta_\infty}\right)}{\partial Z} = \frac{\partial\sigma_{\text{ref}}\left(X(x), \frac{Y(y)}{\beta_\infty}, \frac{Z(z)}{\beta_\infty}\right)}{\partial z}\frac{\partial z}{\partial Z} = \frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial z}\frac{\partial z}{\partial Z} = \frac{1}{\beta_\infty}\frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial z} \tag{19.112}$$

$$\boxed{\left.\frac{\partial\sigma_{\text{ref}}^\star}{\partial Z}\right|_{P_C^I} = \frac{1}{\beta_\infty}\left.\frac{\partial\sigma_{\text{ref}}(x,y,z)}{\partial z}\right|_{P_C^I}} \tag{19.113}$$

Formally identical formulas are also valid for the actual wing surface:

$$\boxed{\left.\frac{\partial\sigma^\star}{\partial X}\right|_{P_C^I} = \left.\frac{\partial\sigma(x,y,z)}{\partial x}\right|_{P_C^I}} \tag{19.114}$$

$$\boxed{\left.\frac{\partial\sigma^\star}{\partial Y}\right|_{P_C^I} = \frac{1}{\beta_\infty}\left.\frac{\partial\sigma(x,y,z)}{\partial y}\right|_{P_C^I}} \tag{19.115}$$

$$\left.\frac{\partial \sigma^\star}{\partial Z}\right|_{P_C^I} = \frac{1}{\beta_\infty} \left.\frac{\partial \sigma\left(x,y,z\right)}{\partial z}\right|_{P_C^I} \tag{19.116}$$

Substituting equations 19.109, 19.111, and 19.113 into equations 19.106 and 19.107:

$$\cos\gamma_I^\star = \cos\gamma_I \tag{19.117}$$

$$\sin\gamma_I^\star = \sin\gamma_I \tag{19.118}$$

The findings of equations 19.117 and 19.118 were expected because the used scaling factors for y and z are the same (see equation 19.78) and because the dihedral angles are measured in the $y-z$ or $Y-Z$ planes. Substituting equations 19.114, 19.115, 19.116, 19.117, and 19.118 into equation 19.105:

$$\left.\frac{\mathrm{d}c^I}{\mathrm{d}X}\right|_{P_C^I} = \frac{\left.\frac{\partial\sigma}{\partial x}\right|_{P_C^I}}{\left(\frac{1}{\beta_\infty}\left.\frac{\partial\sigma}{\partial y}\right|_{P_C^I}\sin\gamma_I - \frac{1}{\beta_\infty}\left.\frac{\partial\sigma}{\partial z}\right|_{P_C^I}\cos\gamma_I\right)} \tag{19.119}$$

or

$$\left.\frac{\mathrm{d}c^I}{\mathrm{d}X}\right|_{P_C^I} = \beta_\infty \left.\frac{\mathrm{d}c^I_{\mathrm{pg}}}{\mathrm{d}x}\right|_{P_C^I} \tag{19.120}$$

where equation 19.100 was used.

Direct inspection of equations 19.82 and 19.84 implies that

$$w^{\mathrm{ss\,dl}}_{\mathrm{nw}\,I} = \frac{1}{V_\infty}\left.\frac{\partial\phi^{\mathrm{inc}}}{\partial Z}\right|_{P_C^I} = \frac{1}{V_\infty}\left.\frac{\partial\left(\mu_\phi\phi^{\mathrm{com}}\right)}{\partial z}\frac{\partial z}{\partial Z}\right|_{P_C^I} = \frac{1}{V_\infty}\left.\frac{\mu_\phi}{\mu_z}\frac{\partial\phi^{\mathrm{com}}}{\partial z}\right|_{P_C^I} = w^{\mathrm{ss\,dl}}_{\mathrm{nw\,pg}\,I} \tag{19.121}$$

From equation 19.80 the pressure jump $\Delta\underline{\boldsymbol{P}}^{\mathrm{ss\,dl}}$ of the transformed plane can be expressed as a function of the pressure $\Delta\underline{\boldsymbol{P}}^{\mathrm{ss\,dl}}_{\mathrm{pg}}$:

$$\Delta\underline{\boldsymbol{P}}^{\mathrm{ss\,dl}} = \beta_\infty\Delta\underline{\boldsymbol{P}}^{\mathrm{ss\,dl}}_{\mathrm{pg}} \tag{19.122}$$

Substituting equations 19.121 and 19.122 into equation 19.83:

$$\boldsymbol{A}^{\mathrm{vl}}\cdot\beta_\infty\Delta\underline{\boldsymbol{P}}^{\mathrm{ss\,dl}}_{\mathrm{pg}} = \boldsymbol{w}^{\mathrm{ss\,dl}}_{\mathrm{nw\,pg}} \tag{19.123}$$

Which is rewritten as follows:

$$\boldsymbol{A}^{\mathrm{vl}}_{\mathrm{pg}}\cdot\Delta\underline{\boldsymbol{P}}^{\mathrm{dl}}_{\mathrm{pg}} = \boldsymbol{w}^{\mathrm{ss\,dl}}_{\mathrm{nw\,pg}} \tag{19.124}$$

Equation 19.124 is formally identical to equation 19.81. This was achieved in the passage from equation 19.123 to equation 19.124 by selecting the *aerodynamic influence matrix for the compressible case* to be equal to the aerodynamic influence matrix of the transformed space multiplied by β_∞. In other words, we have shown that

$$\boldsymbol{A}^{\mathrm{vl}}_{\mathrm{pg}} = \beta_\infty\boldsymbol{A}^{\mathrm{vl}} \tag{19.125}$$

Thus, in the Doublet Lattice Method (discussed later in more detail in chapter 27) the steady part is done by transforming the space x, y, z into X, Y, Z so that matrix $\boldsymbol{A}^{\mathrm{vl}}$ can be easily evaluated with the *incompressible* Vortex Lattice Method. The Prandtl-Glauert compressibility correction is summarized in Figure 19.7.

19.9 Competency Questions

Question # 1 Consider a nonplanar wing reference surface. Sketch in an example a *sending panel* J and a *receiging panel* I. Depict the horshoe vortices on the 2 panels. Indicates the control and load points and where they are located.

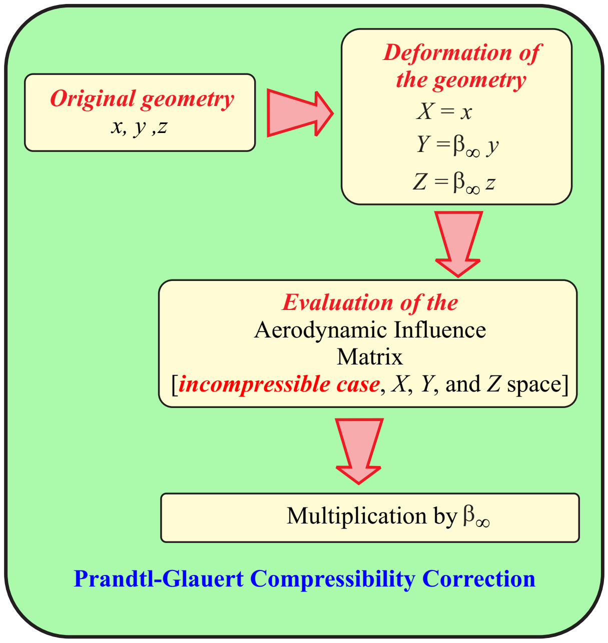

Figure 19.7: Procedure used to formulate the Vortex Lattice Method for steady compressible fluid.

Question # 2 What is the *Vortex Lattice Method* and what does it accomplish?

Question # 3 Given a receiving panel I, how is the *normalwash* calculated?

Question # 4 What is the *aerodynamic influence coefficient*?

Question # 5 What is the *aerodynamic influence matrix*?

Question # 6 Assume to have a wing that is symmetric with respect to the $x - z$ plane. The freestream velocity is directed along x. How is the symmetry is effectively imposed when the Vortex Lattice Method is applied?

Question # 7 Explain how you formulate the Vortex Lattice Method in terms of pressure.

Question # 8 Explain how the Prandtl-Glauert compressibility correction is obtained.

Question # 9 The aerodynamic influence matrix for the compressible case is $\boldsymbol{A}^{\text{vl}}_{\text{pg}}$. The corresponding matrix for the *incompressible* case is $\boldsymbol{A}^{\text{vl}}$. How are $\boldsymbol{A}^{\text{vl}}_{\text{pg}}$ and $\boldsymbol{A}^{\text{vl}}$ related?

Question # 10 The vortex lattice equation is the following:

$$\boldsymbol{A}^{\text{vl}}_{\text{pg}} \cdot \Delta \underline{\boldsymbol{P}}^{\text{dl}}_{\text{pg}} = \boldsymbol{w}^{\text{dl}}_{\text{nw pg}}$$

Explain what the various terms mean.

Corrections to

Chapter 27

Substituting equation 27.34 into equation 27.31:

$$\tilde{w}^{\mathrm{dl}}_{\mathrm{Dnw}\,j\,I}\left(\underline{x}_I, \underline{y}_I, \underline{z}_I, \mathbb{I}\omega\right) = \sum_{J=1}^{N}\left(\frac{\Delta x_J}{8\pi}\fint_{-e_J}^{+e_J} K^{\mathrm{ui}}\left(\underline{x}_I, \underline{y}_I, \underline{z}_I, \underline{\xi}_J = \mu_J \tan\Lambda_J, \underline{\eta}_J = \mu_J, \mathbb{I}\omega\right)\,\mathrm{d}\mu_J + A^{\mathrm{vl}}_{\mathrm{pg}\,I\,J}\right)\Delta\underline{\tilde{\mathscr{P}}}^{\mathrm{dl}}_{j\,J}\left(\mathbb{I}\omega\right) \tag{27.35}$$

Notice that since on the RHS of equation 27.35 the unsteady normalized pressure (in the Fourier domain) is assumed constant, we have only 1 unknown per sending panel. Thus, to completely define the problem, it is deduced that the LHS containing the normalwash, has to be evaluated in a single location on panel I. Thus, with regards to the Doublet Lattice Method we have the *assumption* that the normalwash is imposed in a single point on the receiving panel. In particular, it is imposed on the *control point* P^I_C. Thus, the coordinates $\underline{x}_I$, $\underline{y}_I$, $\underline{z}_I$ have to be the coordinates (in the frame placed on the load point of the sending panel J) of the control point. From the point of view of the solution of the system of integral equations, the control points of receiving panels have the mathematical meaning of *collocation points*.

The integral appearing in equation 27.35 can be written by taking into account equation 27.23 (the explicit arguments in the functions are not shown for brevity, but since the integrals are evaluated only at the doublet lines in all the formulas of the kernel it has to be understood $\underline{\xi}_J = \mu_J \tan\Lambda_J, \underline{\eta}_J = \mu_J$):

$$\frac{\Delta x_J}{8\pi}\fint_{e_J}^{e_J} K^{\mathrm{ui}}\,\mathrm{d}\mu_J = \frac{\Delta x_J}{8\pi}\fint_{-e_J}^{+e_J}\left[\frac{\left(e^{-\mathbb{I}\omega\frac{x_0}{V_\infty}}K_1 - K_{10}\right)T_1}{r^2} + \frac{\left(e^{-\mathbb{I}\omega\frac{x_0}{V_\infty}}K_2 - K_{20}\right)T_2^\star}{r^4}\right]\mathrm{d}\mu_J \tag{27.36}$$

Equation 27.36 is formally simplified as follows:

$$\frac{\Delta x_J}{8\pi}\fint_{-e_J}^{+e_J} K^{\mathrm{ui}}\,\mathrm{d}\mu_J = D^{\mathrm{ui}}_{1IJ} + D^{\mathrm{ui}}_{2IJ} \tag{27.37}$$

where the following definitions have been made:

$$D^{\mathrm{ui}}_{1IJ} = \frac{\Delta x_J}{8\pi}\fint_{-e_J}^{+e_J}\frac{\left(e^{-\mathbb{I}\omega\frac{x_0}{V_\infty}}K_1 - K_{10}\right)T_1}{r^2}\mathrm{d}\mu_J \tag{27.38}$$

$$D^{\mathrm{ui}}_{2IJ} = \frac{\Delta x_J}{8\pi}\fint_{-e_J}^{+e_J}\frac{\left(e^{-\mathbb{I}\omega\frac{x_0}{V_\infty}}K_2 - K_{20}\right)T_2^\star}{r^4}\mathrm{d}\mu_J \tag{27.39}$$

D^{ui}_{1IJ} and D^{ui}_{IJ} are called *normalwash factors*. Notice that all terms within the integrals of equations 27.38 and 27.39 depend on both panels I and J. Thus, both D^{ui}_{1IJ} and D^{ui}_{2IJ} depend on those panels.

Introducing the definition $D_{IJ} := D^{\mathrm{ui}}_{1IJ} + D^{\mathrm{ui}}_{2IJ} + A^{\mathrm{vl}}_{\mathrm{pg}\,I\,J}$, the system of discretized integral equations (see equation 27.35) is rewritten as presented below:

$$\tilde{w}^{\mathrm{dl}}_{\mathrm{Dnw}\,j\,I}\left(\underline{x}_I, \underline{y}_I, \underline{z}_I, , \mathbb{I}\omega\right) = \sum_{J=1}^{N}\left(D^{\mathrm{ui}}_{1IJ} + D^{\mathrm{ui}}_{2IJ} + A^{\mathrm{vl}}_{\mathrm{pg}\,I\,J}\right)\Delta\underline{\tilde{\mathscr{P}}}^{\mathrm{dl}}_{j\,J}\left(\mathbb{I}\omega\right) = \sum_{J=1}^{N} D_{IJ}\,\Delta\underline{\tilde{\mathscr{P}}}^{\mathrm{dl}}_{j\,J}\left(\mathbb{I}\omega\right) \tag{27.40}$$

Notice that the following identity is true:

$$[u_1(c_s+\mathbb{I}k_1)+1](c_s-\mathbb{I}k_1)^2 = c_s^2-k_1^2+u_1c_s(c_s^2+k_1^2)-\mathbb{I}k_1(2c_s+u_1(c_s^2+k_1^2)) \tag{27.75}$$

Substituting equation 27.75 into equation 27.74 and using the result to simplify the integral of equation 27.73, the expression for J_0 (see equation 27.70) takes the following final form:

$$J_0(u_1,k_1)=\sum_{s=1}^{n}a_s e^{-c_s u_1}\frac{c_s^2-k_1^2+u_1c_s(c_s^2+k_1^2)-\mathbb{I}k_1\left[2c_s+u_1(c_s^2+k_1^2)\right]}{(c_s^2+k_1^2)^2} \tag{27.76}$$

Another possible expression for the calculation of integral I_2 is proposed here:

$$\begin{aligned}3I_2 = &\left[\left(2-\frac{u_1(2u_1^2+3)}{(1+u_1^2)^{\frac{3}{2}}}\right)+\mathbb{I}k_1\left(2u_1-\frac{2u_1^2+1}{(1+u_1^2)^{\frac{1}{2}}}\right)\right.\\ &\left.+\;k_1^2e^{+\mathbb{I}k_1u_1}\int_{u_1}^{\infty}e^{-\mathbb{I}k_1u}\left(2u-\frac{2u^2+1}{(1+u^2)^{\frac{1}{2}}}\right)\mathrm{d}u\right]e^{-\mathbb{I}k_1u_1}\end{aligned} \tag{27.77}$$

The demonstration of equation 27.77 is based on 2 integrations by parts and additional algebraic operations (see Appendix R). An advantage of equation 27.77 is that only an integral needs to be evaluated.

- *Case # 2*: $u_1 < 0$

  following the same logic adoped in the derivation of equation 27.66 , we have the following formula:

$$I_2(u_1,k_1)=2\mathbb{R}\mathrm{e}\left[I_2(0,k_1)\right]-\mathbb{R}\mathrm{e}\left[I_2(-u_1,k_1)\right]+\mathbb{I}\cdot\mathbb{I}\mathrm{m}\left[I_2(-u_1,k_1)\right] \tag{27.78}$$

27.7 Final Assembly of the Equations

Key relation is equation 27.40 repeated below with a simplified notation:

$$\tilde{w}^{\mathrm{dl}}_{\mathrm{Dnw}\,j\,I}(\mathbb{I}\omega)=\sum_{J=1}^{N}D_{IJ}(\mathbb{I}\omega)\,\Delta\underline{\tilde{\mathscr{P}}}^{\mathrm{dl}}_{j\,J}(\mathbb{I}\omega) \tag{27.79}$$

where

$$D_{IJ}(\mathbb{I}\omega):=D^{\mathrm{ui}}_{1IJ}(\mathbb{I}\omega)+D^{\mathrm{ui}}_{2IJ}(\mathbb{I}\omega)+A^{\mathrm{vl}}_{\mathrm{pg}\,IJ} \tag{27.80}$$

For practical computer programming, equation 27.79 is written in matrix form:

$$\tilde{\boldsymbol{w}}^{\mathrm{dl}}_{\mathrm{Dnw}\,j}(\mathbb{I}\omega)=\boldsymbol{D}(\mathbb{I}\omega)\cdot\Delta\underline{\tilde{\boldsymbol{P}}}^{\mathrm{dl}}_{j}(\mathbb{I}\omega) \tag{27.81}$$

where $\tilde{\boldsymbol{w}}^{\mathrm{dl}}_{\mathrm{Dnw}\,j}(\mathbb{I}\omega)$ and $\Delta\underline{\tilde{\boldsymbol{P}}}^{\mathrm{dl}}_{j}(\mathbb{I}\omega)$ have both N entries and $\boldsymbol{D}(\mathbb{I}\omega)$ is a $N\times N$ matrix. $\boldsymbol{D}$ is called *aerodynamic influence coefficient matrix*[5] of the Doublet Lattice Method. Notice that given an aerodynamic mesh, $\boldsymbol{D}$

[5]Commerical codes may use a different terminology and attribute the definition of aerodynamic influence coefficient matrix to a different quantity.

is *invariant* with respect to the used mode shape. That can be exploited and makes DLM particularly competitive from a computational perspective and very effective in the preliminary design phases.

It is important to observe that $\Delta\underline{\tilde{\boldsymbol{P}}}_j^{\mathrm{dl}}(\mathbb{I}\omega)$ is an array that contains the normalized Fourier-transformed pressures over every aerodynamic panel. That is an approximation for the function $\Delta\tilde{\mathscr{P}}_j^{\mathrm{dl}}(\boldsymbol{S},\mathbb{I}\omega)$ that should in principle be spatially variable and distributed over the reference aerodynamic surface. However, the normalized Fourier-transformed pressure jump *is not the actual final output of the Doublet Lattice Method.*

27.8 Output of the Doublet Lattice Method: Normalized Fourier-Transformed Generalized Aerodynamic Force Matrix

The product delivered by the DLM is the *normalized Fourier-transformed generalized aerodynamic force matrix* $\tilde{\boldsymbol{A}}^{\mathrm{dl}}$, which is defined as follows:

$$\tilde{\boldsymbol{A}}^{\mathrm{dl}} := \frac{1}{\frac{1}{2}\rho_\infty V_\infty^2}\tilde{\boldsymbol{A}} \tag{27.82}$$

The entries $\tilde{A}_{ij}(\mathbb{I}\omega)$ of the Fourier-transformed generalized aerodynamic force matrix $\tilde{\boldsymbol{A}}$ can be deduced from equation 25.60:

$$\tilde{A}_{ij}(\mathbb{I}\omega) = \int\limits_{\mathrm{wing}} \Delta\tilde{\mathscr{P}}_j^{\mathrm{pirf}}(\boldsymbol{S},\mathbb{I}\omega)\,\varphi_{\mathrm{D}i}^{n}(\boldsymbol{S})\,\mathrm{d}A \tag{27.83}$$

where the pressure impulse response function $\Delta\tilde{\mathscr{P}}_j^{\mathrm{pirf}}$ is obtained by dividing the pressure jump $\Delta\tilde{\mathscr{P}}_j$ of a mode impulsively applied by the unit displacement impulse $\mathscr{U}$, see equation 25.53):

$$\Delta\tilde{\mathscr{P}}_j^{\mathrm{pirf}}(\boldsymbol{S},\mathbb{I}\omega) = \frac{\Delta\tilde{\mathscr{P}}_j(\boldsymbol{S},\mathbb{I}\omega)}{\mathscr{U}} \tag{27.84}$$

Substituting equation 27.84 into equation 27.83:

$$\tilde{A}_{ij}(\mathbb{I}\omega) = \int\limits_{\mathrm{wing}} \frac{\Delta\tilde{\mathscr{P}}_j(\boldsymbol{S},\mathbb{I}\omega)}{\mathscr{U}}\varphi_{\mathrm{D}i}^{n}(\boldsymbol{S})\,\mathrm{d}A \tag{27.85}$$

(Notice that this is more a formal operation rather than a practical implementation because numerically it is $\mathscr{U} = 1$.)

From the definition of equation 27.82 and from the calculation made in equation 27.85 it is deduced that the entries of $\tilde{\boldsymbol{A}}^{\mathrm{dl}}$ are calculated as follows:

$$\tilde{A}_{ij}^{\mathrm{dl}}(\mathbb{I}\omega) = \frac{1}{\frac{1}{2}\rho_\infty V_\infty^2}\int\limits_{\mathrm{wing}} \frac{\Delta\tilde{\mathscr{P}}_j(\boldsymbol{S},\mathbb{I}\omega)}{\mathscr{U}}\varphi_{\mathrm{D}i}^{n}(\boldsymbol{S})\,\mathrm{d}A = \frac{1}{\mathscr{U}}\int\limits_{\mathrm{wing}} \Delta\tilde{\mathscr{P}}_j^{\mathrm{dl}}(\boldsymbol{S},\mathbb{I}\omega)\,\varphi_{\mathrm{D}i}^{n}(\boldsymbol{S})\,\mathrm{d}A \tag{27.86}$$

where the following definition has been introduced:

$$\Delta\tilde{\mathscr{P}}_j^{\mathrm{dl}} := \frac{\Delta\tilde{\mathscr{P}}_j(\boldsymbol{S},\mathbb{I}\omega)}{\frac{1}{2}\rho_\infty V_\infty^2} \tag{27.87}$$

Equation 27.86 is approximated as follows:

$$\tilde{A}_{ij}^{\mathrm{dl}}(\mathbb{I}\omega) \approx \frac{1}{\mathscr{U}}\sum_{I=1}^{N}\left(\Delta\underline{\tilde{\mathscr{P}}}_{j\,I}^{\mathrm{dl}}\cdot A_I\right){}_I\phi_i^{I\,n\,T} = \frac{1}{\mathscr{U}}\sum_{I=1}^{N}\left(\Delta\underline{\tilde{\mathscr{P}}}_{j\,I}^{\mathrm{dl}}\cdot(2e_I\cdot\Delta x_I)\right){}_I\phi_i^{I\,n\,T} \tag{27.88}$$

where A_I is the area of the I^{th} aerodynamic panel, and ${}_I\phi_i^{I\,n\,T}$ is the component of the "displacement" of the i^{th} mode perpendicular to the aerodynamic panel's surface evaluated at the *load point*. The panel is located

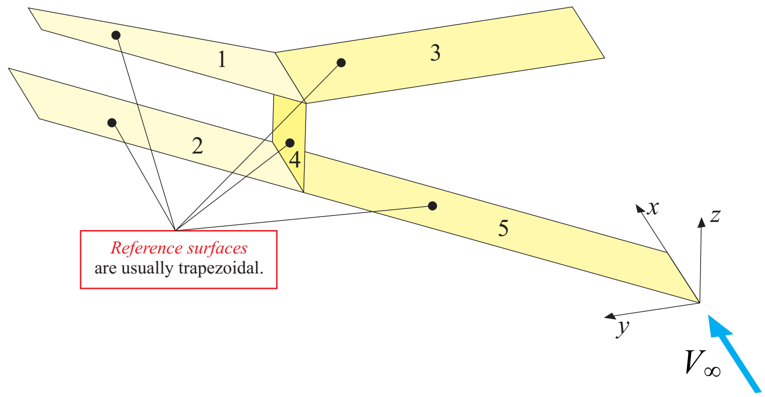

Figure 27.7: Geometry of a generic nonplanar wing: selection of reference surfaces.

by using a different formalism for consitency with the splining (i.e., interface between aerodynamic and structura meshes, see chapter 29):

$$\tilde{w}^{\mathrm{dl}}_{\mathrm{Dnw}\,j\,J} = \mathscr{U}\left(\frac{\mathbb{I}\omega}{V_\infty}\,{}_c\phi_j^{J\,n\,S} + \frac{\partial\,{}_c\phi_j^{J\,n\,S}}{\partial x}\right) \tag{27.92}$$

Discretized normalized Fourier−transformed normalwash

where ${}_c\phi_j^{J\,n\,S}$ indicates the "displacement" due to the j^{th} mode (see Figure 27.6). It is evaluated at the control point and is taken in the direction perpendicular to the surface of the J^{th} panel, which is located on a trapezoidal wing segment of identity S. The quantity $\frac{\partial\,{}_c\phi_j^{J\,n\,S}}{\partial x}$ indicates the slope of the j^{th} mode evaluated at the control point of the same panel (see Figure 27.6). (The reader should observe that the reduced frequency could be easily introduced here; that is not pursued yet to make an easier discussion in following chapters on the stability properties.)

Substituting equation 27.92 into equation 27.91:

$$\tilde{A}^{\mathrm{dl}}_{ij}\left(\mathbb{I}\omega\right) \approx \sum_{I=1}^{N}\sum_{J=1}^{N} \underbrace{\left(\frac{\mathbb{I}\omega}{V_\infty}\,{}_c\phi_j^{J\,n\,S} + \frac{\partial\,{}_c\phi_j^{J\,n\,S}}{\partial x}\right)\cdot{}_I\phi_i^{I\,n\,T}}_{\text{mode } dependent} \cdot \underbrace{D_{IJ}^{-1}\left(\mathbb{I}\omega\right)\cdot\left(2e_I\cdot\Delta x_I\right)}_{\text{mode } invariant} \tag{27.93}$$

Output of the Doublet Lattice Method

27.9 Practical Steps and Guidelines Used in the Implementation of the Doublet Lattice Method

In general a highly nonplanar wing system is discretized with a series of trapezoidal *wing segments*. They represent the *reference surface* (see Figure 27.7). If there is a control surface such as a flap, a wing segment has to be attributed to it as well. Figure 27.8 presents an example of mesh over one of the wing segments. As clearly seen, all the elements are organized into strips parallel to the freestream velocity V_∞. Thus, the mesh is acceptable. The same cannot be said for Figure 27.9 where the mesh over one of wing segment shows that the elements are not properly aligned in the direction of V_∞.

This rule should also be applied for wing segment on the same plane (or almost on the same plane). An example is shown in Figure 27.10 (compare it with Figure 27.11, which shows an acceptable mesh for the wing-tail system). Another important aspect is the case of panels on the same wing segment or in any case on the same plane. In fact, since $T_2^\star = 0$ when $\underline{z}_I = 0$ (see equation 27.16), there is no need to calculate the

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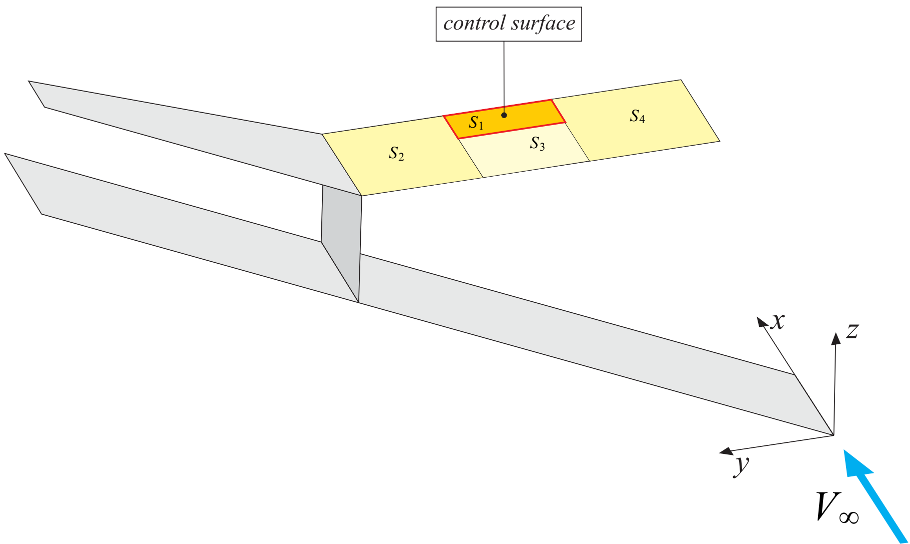

Figure 29.3: An example of wing segments in the case of presence of a control surface.

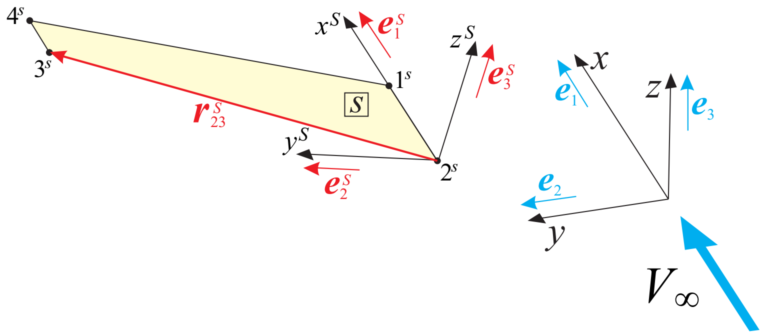

Figure 29.4: Geometry of a wing segment S.

- Each wing segment must be planar and with no angle of attack
- The edges must taken into account that the trapezoidal aerodynamic panels of the Doublet Lattice Method must have 2 edges always parallel to the x direction. The cases of both Figures 29.2 and 29.3 meet that constraint.

The practical steps of the procedure are now discussed. With reference to Figure 29.2, consider wing segment S. The points which identify the wing segments are indicated with 1^S, 2^S, 3^S, and 4^S. The numbering of the points can be done for example by having point 1^S downstream point 2^S. Similarly, point 4^S can be selected downstream point 3^S. A local coordinate system can be created. In the example of Figure 29.2, the coordinate system is centered on point 2^S, but it is not mandatory and other choices are *acceptable*. The axis x^S *must* be parallel to the global axis x. This is a consequence of the fact that the wing segments cannot have any angle of attack. The axis z^S is perpendicular to the wing segment S. The y^S axis is a consequence. In practice the local coordinate system described above can be easily obtained by identifying first x^S and its relative unit vector $\boldsymbol{e}_1^S$. Then (see Figure 29.4), taking the cross product between $\boldsymbol{e}_1^S$ and a position vector connecting points 2^S and 3^S, the direction normal to the wing segment is identified. This is now presented in detail. Let the *global coordinates* of the points be identified as follows:

$$1^S \equiv (x_{1^S}, y_{1^S}, z_{1^S}) \quad 2^S \equiv (x_{2^S}, y_{2^S}, z_{2^S}) \quad 3^S \equiv (x_{3^S}, y_{3^S}, z_{3^S}) \quad 4^S \equiv (x_{4^S}, y_{4^S}, z_{4^S}) \tag{29.35}$$

The position vector connecting points 2^S and 3^S is, in the global coordinate system, the following (see Figure 29.4):

$$\boldsymbol{r}_{23}^S = (x_{3^S} - x_{2^S})\,\boldsymbol{e}_1 + (y_{3^S} - y_{2^S})\,\boldsymbol{e}_2 + (z_{3^S} - z_{2^S})\,\boldsymbol{e}_3 \tag{29.36}$$

or

$$\frac{\partial\, {}_cK^S_{Iq}}{\partial x^S} = 2\left({}_cx^S_I - x^S_q\right)\left[\ln\left({}_cr^S_{Iq}\right)^2 + 1\right] \tag{29.67}$$

Notice that when control point and "strong node" coincide it is deduced from equation 29.66 that $\frac{\partial\, {}_cK^S_{Iq}}{\partial x^S} = 0$.

Taking into account equation 29.67 and equation 29.61 it is deduced that the derivatives of the projections of the i^{th} mode in the direction perpendicular to the wing segment S are the following:

$$\begin{bmatrix} \frac{\partial\, {}_c\phi_i^{1\,n\,S}}{\partial x^S} \\ \frac{\partial\, {}_c\phi_i^{2\,n\,S}}{\partial x^S} \\ \frac{\partial\, {}_c\phi_i^{3\,n\,S}}{\partial x^S} \\ \cdots \\ \frac{\partial\, {}_c\phi_i^{N^S\,n\,S}}{\partial x^S} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \frac{\partial\, {}_cK^S_{11}}{\partial x^S} & \frac{\partial\, {}_cK^S_{12}}{\partial x^S} & \frac{\partial\, {}_cK^S_{13}}{\partial x^S} & \cdots & \frac{\partial\, {}_cK^S_{1\,N^S_n}}{\partial x^S} \\ 0 & 1 & 0 & \frac{\partial\, {}_cK^S_{21}}{\partial x^S} & \frac{\partial\, {}_cK^S_{22}}{\partial x^S} & \frac{\partial\, {}_cK^S_{23}}{\partial x^S} & \cdots & \frac{\partial\, {}_cK^S_{2\,N^S_n}}{\partial x^S} \\ 0 & 1 & 0 & \frac{\partial\, {}_cK^S_{31}}{\partial x^S} & \frac{\partial\, {}_cK^S_{32}}{\partial x^S} & \frac{\partial\, {}_cK^S_{33}}{\partial x^S} & \cdots & \frac{\partial\, {}_cK^S_{3\,N^S_n}}{\partial x^S} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \frac{\partial\, {}_cK^S_{N^S1}}{\partial x^S} & \frac{\partial\, {}_cK^S_{N^S2}}{\partial x^S} & \frac{\partial\, {}_cK^S_{N^S3}}{\partial x^S} & \cdots & \frac{\partial\, {}_cK^S_{N^S N^S_n}}{\partial x^S} \end{bmatrix} \begin{bmatrix} a^S_0 \\ a^S_1 \\ a^S_2 \\ F^S_1 \\ F^S_2 \\ F^S_3 \\ \cdots \\ F^S_{N^S_n} \end{bmatrix} \tag{29.68}$$

The derivatives need to be understood evaluated at the control points of the aerodynamic panels present on wing segment S.

Equation 29.68 is written in compact form introducing arrays and matrices:

$$\frac{\partial\, {}_c\boldsymbol{\phi}_i^{n\,S}}{\partial x^S} = \boldsymbol{D}^S \cdot \boldsymbol{F}^S \tag{29.69}$$

Notice that the array $\frac{\partial\, {}_c\boldsymbol{\phi}_i^{n\,S}}{\partial x^S}$ containing the derivatives has dimension $N^S \times 1$, $\boldsymbol{D}^S$ has dimension $N^S \times (N^S_n + 3)$ and $\boldsymbol{F}^S$ has dimension $(N^S_n + 3) \times 1$. Substituting equation 29.57 in equation 29.69:

$$\frac{\partial\, {}_c\boldsymbol{\phi}_i^{n\,S}}{\partial x^S} = \boldsymbol{D}^S \cdot \boldsymbol{F}^S = \boldsymbol{D}^S \cdot \left[\boldsymbol{G}^S\right]^{-1} \cdot {}^\star\boldsymbol{\phi}_i^{n\,S} \tag{29.70}$$

Observing that the first three rows of ${}^\star\boldsymbol{\phi}_i^{n\,S}$ are zeros (see its definition introduced in equation 29.55), it is possible *to eliminate the first three columns* of the matrix $\left[\boldsymbol{G}^S\right]^{-1}$ without changing the result. Defining $\boldsymbol{S}^S$, the matrix $\left[\boldsymbol{G}^S\right]^{-1}$ with the first three columns eliminated, and defining $\boldsymbol{\phi}_i^{n\,S}$, the vector ${}^\star\boldsymbol{\phi}_i^{n\,S}$ without the first three rows, equation 29.65 can be rewritten as shown below:

$$\boxed{\frac{\partial\, {}_c\boldsymbol{\phi}_i^{n\,S}}{\partial x^S} = \boldsymbol{D}^S \cdot \boldsymbol{S}^S \cdot \boldsymbol{\phi}_i^{n\,S}} \tag{29.71}$$

Notice that $\boldsymbol{S}^S$ has dimension $(N^S_n + 3) \times N^S_n$ and $\frac{\partial\, {}_c\boldsymbol{\phi}_i^{n\,S}}{\partial x^S}$ has dimension $N^S \times 1$.

29.3.6 Calculating the Mode Shape at the Load Points of the Aerodynamic Mesh

To calculate the entries of the normalized Fourier-transformed generalized aerodynamic force matrix of equation 29.34 we need to calculate the value of the component ${}_l\phi_i^{I\,n\,S}$ of the mode shape perpendicular to the wing segment in correspondence of the *load point* of the generic aerodynamic panel located on that wing segment. Thus, the same procedure used for the control points is now adopted for the load points. We introduce a notation similar to the one previously discussed (the left subscript c is replaced now by l to emphasize "load point").

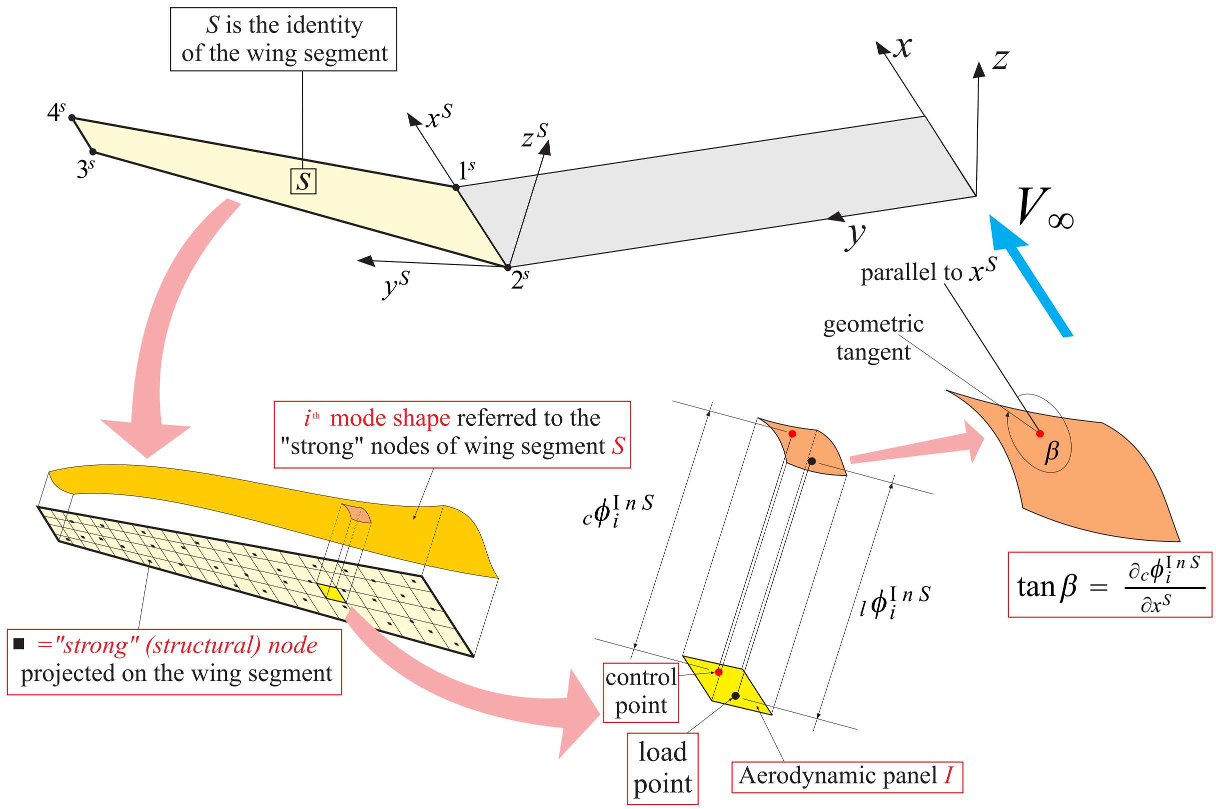

Figure 29.5: Wing segment S, aerodynamic mesh, panel I, and meaning of ${}_c\phi_i^{I\,n\,S}$, ${}_l\phi_i^{I\,n\,S}$, and $\frac{\partial\,{}_c\phi_i^{I\,n\,S}}{\partial x^S}$.

29.5 The Aerodynamic Influence Coefficient Matrix in Structural Degrees of Freedom

From a computational perspective, it may be convenient to express equation 29.34 in terms of products of matrices, with the intent to exploit that for a given aerodynamic and structural meshes the transformation matrices do not change, taht is, they are *invariant with respect to the mode shape identities*.

Once the spline coefficients have been determined for each wing segment (see equation 29.57), we can use them to build a *known* transformation matrix $\boldsymbol{B}_J$ (which has 1 row and N_{DOF} columns) that for any given mode shape $\boldsymbol{\phi}_j$ allows one to find the mode "displacement" ${}_c\phi_j^{J\,n\,S}$ perpendicular to the aerodynamic panel at the control point of the J^{th} panel:

$$ {}_c\phi_j^{J\,n\,S} = \boldsymbol{B}_J \cdot \boldsymbol{\phi}_j = \boldsymbol{\phi}_j^\top \cdot \boldsymbol{B}_J^\top \tag{29.85}$$

(Notice that the second equality of equation 29.85 holds because ${}_c\phi_j^{J\,n\,S}$ is a scalar and its transpose returns the same quantity.) Similarly, the derivatives $\frac{\partial\,{}_c\phi_j^{J\,n\,S}}{\partial x}$ of the mode shapes with respect to the direction of the freestream velocity (see equation 29.71 and relative discussion) can be expressed with a known transformation matrix $\boldsymbol{C}_J$ as follows:

$$ \frac{\partial\,{}_c\phi_j^{J\,n\,S}}{\partial x} = \boldsymbol{C}_J \cdot \boldsymbol{\phi}_j = \boldsymbol{\phi}_j^\top \cdot \boldsymbol{C}_J^\top \tag{29.86}$$

$\boldsymbol{C}_J$ has 1 row and N_{DOF} columns as well.

The same logic can be applied for the mode "deflections" ${}_l\phi_i^{I\,n\,T}$ evaluated at the I^{th} load point. That is, from the knowledge of the splining at wing segment level (see equation 29.76) we can build another matrix $\boldsymbol{E}_I$ (which has size $1 \times N_{DOF}$) and express ${}_l\phi_i^{I\,n\,T}$ as function of the finite element mode shape $\boldsymbol{\phi}_i$:

$$ {}_l\phi_i^{I\,n\,T} = \boldsymbol{E}_I \cdot \boldsymbol{\phi}_i = \boldsymbol{\phi}_i^\top \cdot \boldsymbol{E}_I^\top \tag{29.87}$$

Substituting equations 29.85, 29.86, and 29.87 into equation 29.34:

$$\tilde{A}^{\mathrm{dl}}_{ij}(\mathbb{I}\omega) \approx \sum_{I=1}^{N}\sum_{J=1}^{N}\left(\frac{\mathbb{I}\omega}{V_\infty}\boldsymbol{\phi}_j^\top\cdot\boldsymbol{B}_J^\top+\boldsymbol{\phi}_j^\top\cdot\boldsymbol{C}_J^\top\right)\cdot\boldsymbol{E}_I\cdot\boldsymbol{\phi}_i\cdot D_{IJ}^{-1}(\mathbb{I}\omega)\cdot(2e_I\cdot\Delta x_I) \tag{29.88}$$

$\tilde{A}^{\mathrm{dl}}_{ij}(\mathbb{I}\omega)$ is a scalar. Thus, if its transpose is evaluated, the result is not affected. Using this argument, equation 29.88 is rewritten as follows:

$$\tilde{A}^{\mathrm{dl}}_{ij}(\mathbb{I}\omega) \approx \sum_{I=1}^{N}\sum_{J=1}^{N}\left[\left(\frac{\mathbb{I}\omega}{V_\infty}\boldsymbol{\phi}_j^\top\cdot\boldsymbol{B}_J^\top+\boldsymbol{\phi}_j^\top\cdot\boldsymbol{C}_J^\top\right)\cdot\boldsymbol{E}_I\cdot\boldsymbol{\phi}_i\cdot D_{IJ}^{-1}(\mathbb{I}\omega)\cdot(2e_I\cdot\Delta x_I)\right]^\top \tag{29.89}$$

or (notice that $\frac{\mathbb{I}\omega}{V_\infty}$ and $D_{IJ}^{-1}(\mathbb{I}\omega)\cdot(2e_I\cdot\Delta x_I)$ are scalars; thus they can be placed in different positions when a product is performed)

$$\tilde{A}^{\mathrm{dl}}_{ij}(\mathbb{I}\omega) \approx \sum_{I=1}^{N}\sum_{J=1}^{N}\boldsymbol{\phi}_i^\top\cdot\boldsymbol{E}_I^\top\cdot D_{IJ}^{-1}(\mathbb{I}\omega)\cdot(2e_I\cdot\Delta x_I)\cdot\left(\frac{\mathbb{I}\omega}{V_\infty}\boldsymbol{B}_J\cdot\boldsymbol{\phi}_j+\boldsymbol{C}_J\cdot\boldsymbol{\phi}_j\right) \tag{29.90}$$

It is now observed that both $\boldsymbol{\phi}_i^\top$ and $\boldsymbol{\phi}_j$ are independent of the indices I and J. Thus, they can be brought outside the double summation of equation 29.90 and further simplifications are possible:

$$\tilde{A}^{\mathrm{dl}}_{ij}(\mathbb{I}\omega) \approx \boldsymbol{\phi}_i^\top\cdot\left[\sum_{I=1}^{N}\sum_{J=1}^{N}\boldsymbol{E}_I^\top\cdot D_{IJ}^{-1}(\mathbb{I}\omega)\cdot(2e_I\cdot\Delta x_I)\cdot\left(\frac{\mathbb{I}\omega}{V_\infty}\boldsymbol{B}_J+\boldsymbol{C}_J\right)\right]\cdot\boldsymbol{\phi}_j \tag{29.91}$$

The following definition is now introduced:

$$\boldsymbol{A}^{\mathrm{AIC}}(\mathbb{I}\omega) := \sum_{I=1}^{N}\sum_{J=1}^{N}\boldsymbol{E}_I^\top\cdot D_{IJ}^{-1}(\mathbb{I}\omega)\cdot(2e_I\cdot\Delta x_I)\cdot\left(\frac{\mathbb{I}\omega}{V_\infty}\boldsymbol{B}_J+\boldsymbol{C}_J\right) \tag{29.92}$$

$\boldsymbol{A}^{\mathrm{AIC}}(\mathbb{I}\omega)$ is the *aerodynamic influence coefficient matrix in structural degrees of freedom*. It is a squared $N_{DOF}\times N_{DOF}$ matrix and is invariant with respect to the mode shapes, for given aerodynamic and structural meshes. Even if it was not explicitly written in the arguments, $\boldsymbol{A}^{\mathrm{AIC}}(\mathbb{I}\omega)$ also depends on the Mach number because $D_{IJ}^{-1}(\mathbb{I}\omega)$ does.

Using the definition introduced in equation 29.92, we can write (see equation 29.91):

$$\tilde{A}^{\mathrm{dl}}_{ij}(\mathbb{I}\omega) \approx \boldsymbol{\phi}_i^\top\cdot\underbrace{\boldsymbol{A}^{\mathrm{AIC}}(\mathbb{I}\omega)}_{\text{mode } invariant}\cdot\boldsymbol{\phi}_j \tag{29.93}$$

Equation 29.93 is very useful in case large number of aeroelastic simulations need to be performed, because given the aerodynamic and structural models $\boldsymbol{A}^{\mathrm{AIC}}(\mathbb{I}\omega)$ can be calculated for a set of frequencies and Mach numbers and reused to perform thousands of simulations. This is the case, for example, of external stores. They have a relevant mass and, thus, alter the mode shapes of the entire airplane. The worst case scenario (i.e., the instability occurs at lower speed) needs to be found for all possible combinations and many analyses need to be performed.

29.6 Theoretical Differences Between Doublet Lattice Method and Other Potential Flow Methods

The Doublet Lattice Method is based on the main assumption of potential flow. That is, the fluid viscosity is neglected. There exist alternative methods that use the same hypothesis. They present advantages and disadvantages with respect to the Doublet Lattice Method.

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Equation 33.20 is recast in matrix as follows:

$$\dot{\boldsymbol{x}}(t) = \left[\begin{array}{ccccc} \lambda_1 \boldsymbol{W}_1 & \lambda_2 \boldsymbol{W}_2 & \lambda_3 \boldsymbol{W}_3 & ... & \lambda_n \boldsymbol{W}_n \end{array} \right] \cdot \left[\begin{array}{c} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ ... \\ e^{\lambda_n t} \end{array} \right] \tag{33.22}$$

where $\boldsymbol{W}_j$ are the column vectors of $\boldsymbol{W}$:

$$\boldsymbol{W}_j = \left[\begin{array}{ccccc} W_{1j} & W_{2j} & W_{3j} & ... & W_{nj} \end{array} \right]^\top \tag{33.23}$$

so that, by definition of column vectors, matrix $\boldsymbol{W}$ can be written as presented below:

$$\boldsymbol{W} = \left[\begin{array}{ccccc} \boldsymbol{W}_1 & \boldsymbol{W}_2 & \boldsymbol{W}_3 & ... & \boldsymbol{W}_n \end{array} \right] \tag{33.24}$$

Substituting equation 33.24 into equation 33.20, taking the result and substituting it into equation 33.7 and using equation 33.22, the following is obtained:

$$\dot{\boldsymbol{x}}(t) = \left[\begin{array}{ccccc} \lambda_1 \boldsymbol{W}_1 & \lambda_2 \boldsymbol{W}_2 & \lambda_3 \boldsymbol{W}_3 & ... & \lambda_n \boldsymbol{W}_n \end{array} \right] \cdot \left[\begin{array}{c} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ ... \\ e^{\lambda_n t} \end{array} \right] = \boldsymbol{A} \cdot \boldsymbol{x}(t) = \boldsymbol{A} \cdot \boldsymbol{W} \cdot \left[\begin{array}{c} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ ... \\ e^{\lambda_n t} \end{array} \right] \tag{33.25}$$

which is satisfied if

$$\left[\begin{array}{ccccc} \boldsymbol{A} \cdot \boldsymbol{W}_1 - \lambda_1 \boldsymbol{W}_1 & \boldsymbol{A} \cdot \boldsymbol{W}_2 - \lambda_2 \boldsymbol{W}_2 & \boldsymbol{A} \cdot \boldsymbol{W}_3 - \lambda_3 \boldsymbol{W}_3 & ... & \boldsymbol{A} \cdot \boldsymbol{W}_n - \lambda_n \boldsymbol{W}_n \end{array} \right] = \boldsymbol{0} \tag{33.26}$$

or, equivalently

$$\boldsymbol{A} \cdot \boldsymbol{W}_i = \lambda_i \boldsymbol{W}_i \qquad i = 1, 2, 3, ..., n \tag{33.27}$$

or

$$[\boldsymbol{A} - \lambda_i \boldsymbol{I}] \cdot \boldsymbol{W}_i = \boldsymbol{0} \qquad i = 1, 2, 3, ..., n \tag{33.28}$$

Equation 33.28 is the definition of *eigenvectors* of matrix $\boldsymbol{A}$, where the *eigenvalues* are λ_i and are found by setting

$$\det [\boldsymbol{A} - \lambda_i \boldsymbol{I}] = 0 \tag{33.29}$$

At this stage of the discussion, assume that all the eigenvalues are real and distinct. If that is the case, the number of eigenvectors is n. The matrix $\boldsymbol{W}$ defined in equation 33.24 is called *modal matrix* and its columns are the eigenvectors. By definition of eigenvectors, they are not uniquely defined. Thus, after scaling the eigenvectors, the most general solution can be written as follows:

$$\boldsymbol{x}(t) = \left[\begin{array}{ccccc} \eta_1 \boldsymbol{W}_1 & \eta_2 \boldsymbol{W}_2 & \eta_3 \boldsymbol{W}_3 & ... & \eta_n \boldsymbol{W}_n \end{array} \right] \cdot \left[\begin{array}{c} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ ... \\ e^{\lambda_n t} \end{array} \right] \tag{33.30}$$

where the coefficients η_1, η_2 etc. are the *modal coordinates* and need to be found.

Equation 33.30 can be written in a more convenient form:

$$\boldsymbol{x}(t) = \left[\begin{array}{ccccc} \boldsymbol{W}_1 & \boldsymbol{W}_2 & \boldsymbol{W}_3 & ... & \boldsymbol{W}_n \end{array} \right] \cdot \left[\begin{array}{ccccc} e^{\lambda_1 t} & 0 & 0 & ... & 0 \\ 0 & e^{\lambda_2 t} & 0 & ... & 0 \\ 0 & 0 & e^{\lambda_3 t} & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & e^{\lambda_n t} \end{array} \right] \cdot \left[\begin{array}{c} \eta_1 \\ \eta_2 \\ ... \\ \eta_n \end{array} \right] \tag{33.31}$$

33.4.4 Example of Stability Analysis

The second order ordinary differential equation 32.1 was extensively studied (see chapter 32). It is now analyzed again with the approach of considering an *equivalent* Linear Time-Invariant System. That system was obtained in equations 33.7 and 33.8. To study the stability the eigenvalues are first determined:

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} -\lambda & 1 \\ -\dfrac{C}{A} & -\dfrac{B}{A} - \lambda \end{bmatrix} \tag{33.56}$$

The characteristic equation is obtained as follows:

$$\det\left[\boldsymbol{A} - \lambda \boldsymbol{I}\right] = -\lambda\left(-\frac{B}{A} - \lambda\right) - \left(-\frac{C}{A}\right) = 0 \Rightarrow \lambda^2 + \lambda\frac{B}{A} + \frac{C}{A} = 0 \tag{33.57}$$

Equation 33.57 is formally identical to the denominator of the Laplace transform (see equation 32.10). That is not a coincidence. Thus, all the discussion that was conducted on the poles was practically a discussion on the eigenvalues. Since this has now been established, it is possible to gain additional insights. For example, consider the case studied in Figure 32.4 and corresponding to $B = C = 0; A > 0$ (that was a situation of an unstable system). In that case, using equation 33.56, the eigenvector associated with $\lambda_1 = 0$ (which is a double solution) can be determined by using the following definition:

$$\left[\boldsymbol{A} - \lambda_1 \boldsymbol{I}\right] \cdot \boldsymbol{W} = \boldsymbol{0} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_{11} \\ W_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{33.58}$$

Observing that the second row of the matrix of coefficients in equation 33.58 is made of only zeros, the only non-trivial equation that can be written is determined by the first row of the matrix:

$$W_{12} = 0 \tag{33.59}$$

That means the eigenvector is made of any array of the type

$$\begin{bmatrix} W_{11} \\ 0 \end{bmatrix} \tag{33.60}$$

where W_{11} can be freely selected. Setting $W_{11} = 1$, the eigenvector associated with the eigenvalue $\lambda_1 = 0$ is

$$_1\boldsymbol{W} := \boldsymbol{W}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{33.61}$$

which is also the generalized eigenvector of order 1 by definition.

Since the eigenvalue is defective (i.e., there is in this case a “missing” eigenvector), then the generalized eigenvector of order 2 is built by using its definition introduced in equation 33.48:

$$\left[\boldsymbol{A} - \lambda_1 \boldsymbol{I}\right] \cdot \,_2\boldsymbol{W}_1 = \,_1\boldsymbol{W}_1 \tag{33.62}$$

or

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} _2W_{11} \\ _2W_{12} \end{bmatrix} = \begin{bmatrix} _1W_{11} \\ _1W_{12} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{33.63}$$

The second equation of the system 33.63 provides the trivial identity $0 = 0$. The first equation leads to the relationship shown below:

$$0 \cdot \,_2W_{11} + 1 \cdot \,_2W_{12} = 1 \Rightarrow \,_2W_{12} = 1 \tag{33.64}$$

Notice that the first component was not determined. We can pick ${}_2W_{11} = 0$. Thus, the generalized eigenvector is the following:

$$ {}_2\boldsymbol{W}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{33.65} $$

A solution of the system of equations is represented by the generalized eigenvector of order 1 (which is the standard mode). Thus, setting $k = 1$, $j = 1$ (there is only one eigenvalue $\lambda_1 = 0$) and substituting equation 33.61 into equation 33.50:

$$ {}_1\boldsymbol{x}_1 = e^{\lambda_1 t}\; {}_1\boldsymbol{W}_1 = \; {}_1\boldsymbol{W}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{33.66} $$

Now we operate on the generalized eigenvector of order 2. Setting $k = 2$, $j = 1$ (there is only one eigenvalue $\lambda_1 = 0$) and substituting equation 33.65 into equation 33.50:

$$ {}_2\boldsymbol{x}_1 = e^{\lambda_1 t}\left[\, {}_2\boldsymbol{W}_1 + t\; {}_1\boldsymbol{W}_1 \right] = \; {}_2\boldsymbol{W}_1 + t\; {}_1\boldsymbol{W}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{33.67} $$

Since both equations 33.66 and 33.67 are elementary solutions, then a linear combination provides the final solution:

$$ \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \cdot \; {}_1\boldsymbol{x}_1 + c_2 \cdot \; {}_2\boldsymbol{x}_1 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] \tag{33.68} $$

which is equivalent to the following writing:

$$ \begin{cases} x_1(t) = c_1 + c_2 t \\ x_2(t) = c_2 \end{cases} \tag{33.69} $$

In this specific case, the system of equations was actually a second order equation. Thus, if the original definitions (see equation 33.2) are reinstated in equation 33.69, the following is obtained:

$$ \begin{cases} x(t) = c_1 + c_2 t \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \equiv \dot{x}(t) = c_2 \end{cases} \tag{33.70} $$

Notice that first relation of equation 33.70 is the same of equation 32.48 after the initial conditions are used to determine c_1 and c_2.

33.4.5 Linear Time-Invariant Systems Written in the Laplace Domain

The homogeneous case is investigated here. That is, the system of differential equations does not have input terms. Thus, the Linear Time-Invariant System of equation 33.7 of n differential equations is trivially transformed as follows:

$$ \mathscr{L}\left(\dot{\boldsymbol{x}}(t)\right) = \mathscr{L}\left(\boldsymbol{A} \cdot \boldsymbol{x}(t)\right) \tag{33.71} $$

or

$$ s\mathscr{L}\left(\boldsymbol{x}(t)\right) - \boldsymbol{x}_0 = \boldsymbol{A} \cdot \mathscr{L}\left(\boldsymbol{x}(t)\right) \tag{33.72} $$

or

$$ \boxed{s\hat{\boldsymbol{x}}(s) = \boldsymbol{A}\hat{\boldsymbol{x}}(s) + \boldsymbol{x}_0} \tag{33.73} $$

We will write the aeroelastic equations (see chapter 34) in the form that was shown in equation 33.73.

Corrections to

Chapter 35

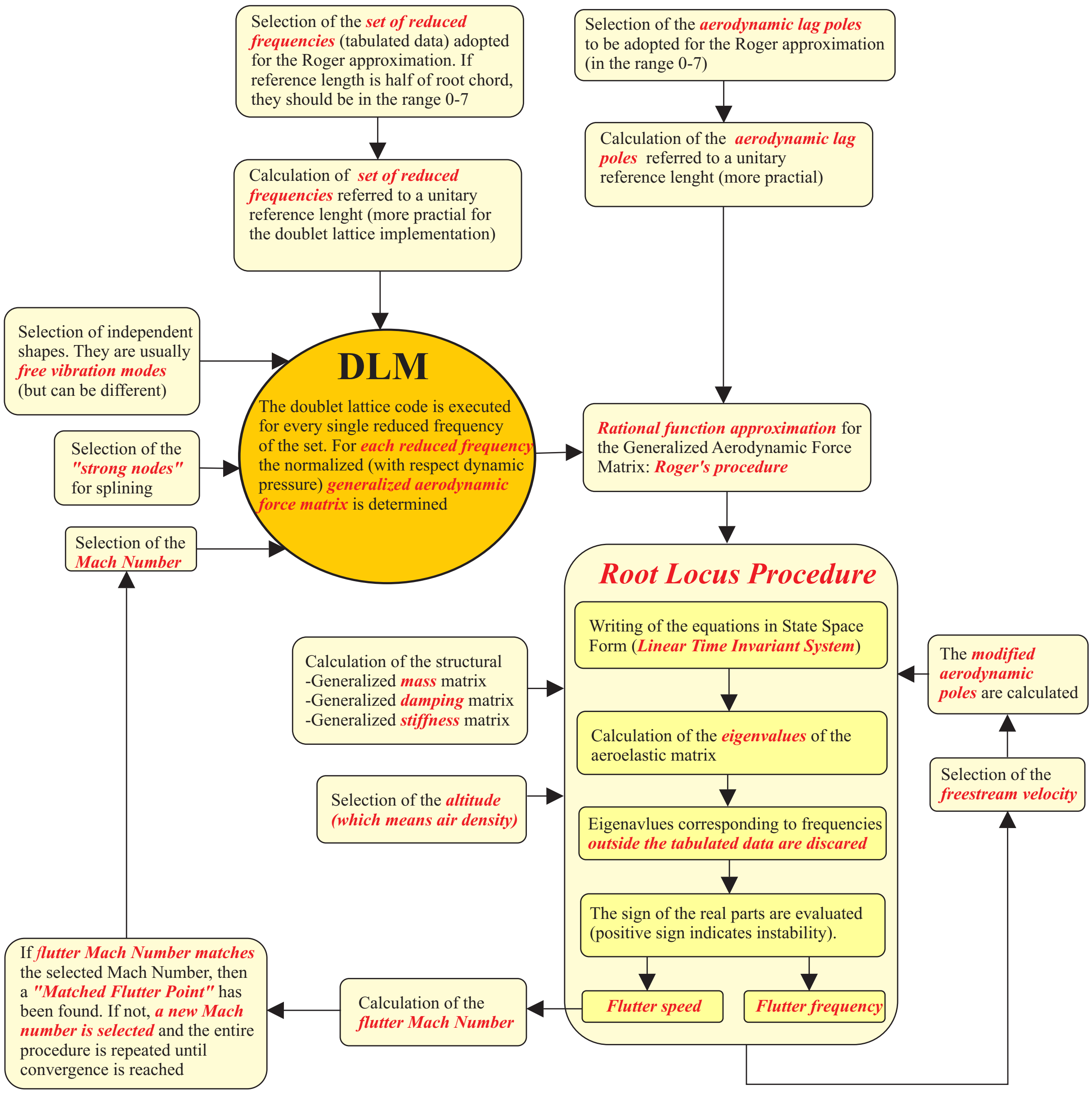

Figure 35.10: Calculation of flutter point by using Doublet Lattice Method, Roger's Procedure, and Root Locus technique.

Corrections to Chapter 36

M_∞, V_∞, ρ_∞, and g) which need to be determined to have a non-trivial solution of equation 36.29. That is, we seek a solution that satisfies equation 36.29 for $\tilde{\boldsymbol{q}} \neq \mathbf{0}$.

The freestream *speed of sound* c_∞ is given by the formula introduced in equation 7.74:

$$c_\infty = \sqrt{\gamma R^\star T_\infty} \tag{36.30}$$

Since T_∞ depends on the altitude, it is deduced that also the speed of sound c_∞ depends on the altitude. Also, recall the definition of Mach number:

$$M_\infty = \frac{V_\infty}{c_\infty} \tag{36.31}$$

We transform equation 36.29 into an eigenvalue problem by writing that expression in terms of only 1 unknown parameter. *First*, the altitude is fixed. Thus, c_∞ and ρ_∞ are known. We also fix the *reduced frequency* k to a value (we need to cover a range of reduced frequencies of possible interest; that can be done by knowing similar airplanes and historical data to estimate where the flutter point might be and which range of reduced frequencies need to be considered). *Second*, M_∞ is fixed. The kernel of the Doublet Lattice Method is now fully determined and, thus, the Fourier-transformed normalized generalized aerodynamic force matrix $\tilde{\boldsymbol{A}}^{\mathrm{dl}}\left(\mathbb{I}\frac{\omega b}{V_\infty}, M_\infty\right)$ is calculated with the DLM software.

The knowledge of the Mach number is also used in equation 36.31 to find the freestream velocity (candidate to be the flutter speed):

$$V_\infty = M_\infty c_\infty \tag{36.32}$$

Equation 36.32 is *not* a priori enforced in the k method. Thus, it is said that we do not have a matched flutter point. This means, from a practical perspective, that the flutter speed obtained with this method violates, in general, the condition presented in equation 36.32.

At this stage, given the fact that we fixed all of the other parameters (see discussion above) the only unknown is the dummy damping g that needs to be added or subtracted to keep having harmonic motion (the solution will determine that information; the reader should keep in mind that only the value of $g = 0$, which is the flutter point, has physical meaning in the framework of the k method). We elaborate equation 36.29 by conceptually considering the aerodynamic loads in the frequency domain as "mass terms". Thus, the portion containing the normalized Fourier-transformed generalized aerodynamic force matrix is first placed next to the generalized mass matrix $\overline{\boldsymbol{M}}$ of the structure as follows:

$$\left[(\mathbb{I}\omega)^2\,\overline{\boldsymbol{M}} - \frac{1}{2}\rho_\infty V_\infty^2 \tilde{\boldsymbol{A}}^{\mathrm{dl}}\left(\mathbb{I}\frac{\omega b}{V_\infty}, M_\infty\right) + (\mathbb{I}\omega)\,\overline{\boldsymbol{C}} + (1+\mathbb{I}g)\,\overline{\boldsymbol{K}}\right]\cdot\tilde{\boldsymbol{q}} = \mathbf{0} \tag{36.33}$$

To group the "mass terms" together, we multiply and divide by ω^2 the aerodynamic part of equation 36.33:

$$\left[(\mathbb{I}\omega)^2\,\overline{\boldsymbol{M}} - \frac{1}{2}\rho_\infty V_\infty^2 \frac{\omega^2}{\omega^2}\tilde{\boldsymbol{A}}^{\mathrm{dl}}\left(\mathbb{I}\frac{\omega b}{V_\infty}, M_\infty\right) + (\mathbb{I}\omega)\,\overline{\boldsymbol{C}} + (1+\mathbb{I}g)\,\overline{\boldsymbol{K}}\right]\cdot\tilde{\boldsymbol{q}} = \mathbf{0} \tag{36.34}$$

Recalling the identity $(\mathbb{I}\omega)^2 = -\omega^2$, equation 36.34 is written as shown below:

$$\left[\left[\overline{\boldsymbol{M}} + \frac{1}{2}\rho_\infty \frac{V_\infty^2}{\omega^2}\tilde{\boldsymbol{A}}^{\mathrm{dl}}\left(\mathbb{I}\frac{\omega b}{V_\infty}, M_\infty\right)\right]\left(-\omega^2\right) + (\mathbb{I}\omega)\,\overline{\boldsymbol{C}} + (1+\mathbb{I}g)\,\overline{\boldsymbol{K}}\right]\cdot\tilde{\boldsymbol{q}} = \mathbf{0} \tag{36.35}$$

We now introduce a new auxiliary *complex variable* λ defined as follows:

$$\lambda := \frac{\mathbb{I}\omega}{\sqrt{1+\mathbb{I}g}} \quad \Rightarrow \quad \mathbb{I}\omega = \lambda\sqrt{1+\mathbb{I}g} \quad \text{and} \quad -\omega^2 = \lambda^2\,(1+\mathbb{I}g) \tag{36.36}$$

Observe from equation 36.36 that at the flutter point (subscript F is used to identify that special condition) it is by definition $g = g_F = 0$ and $\lambda = \lambda_F = \mathbb{I}\omega_F$.

Substituting equation 36.36 into equation 36.35 and evaluating the normalized Fourier-transformed generalized aerodynamic force matrix at the fixed reduced frequency allows one to write the following relationship:

$$\left[\left[\overline{\boldsymbol{M}} + \frac{1}{2}\rho_\infty \frac{V_\infty^2}{\omega^2}\tilde{\boldsymbol{A}}^{\mathrm{dl}}\left(\mathbb{I}k, M_\infty\right)\right]\lambda^2\,(1+\mathbb{I}g) + \lambda\sqrt{1+\mathbb{I}g}\,\overline{\boldsymbol{C}} + (1+\mathbb{I}g)\,\overline{\boldsymbol{K}}\right]\cdot\tilde{\boldsymbol{q}} = \mathbf{0} \tag{36.37}$$

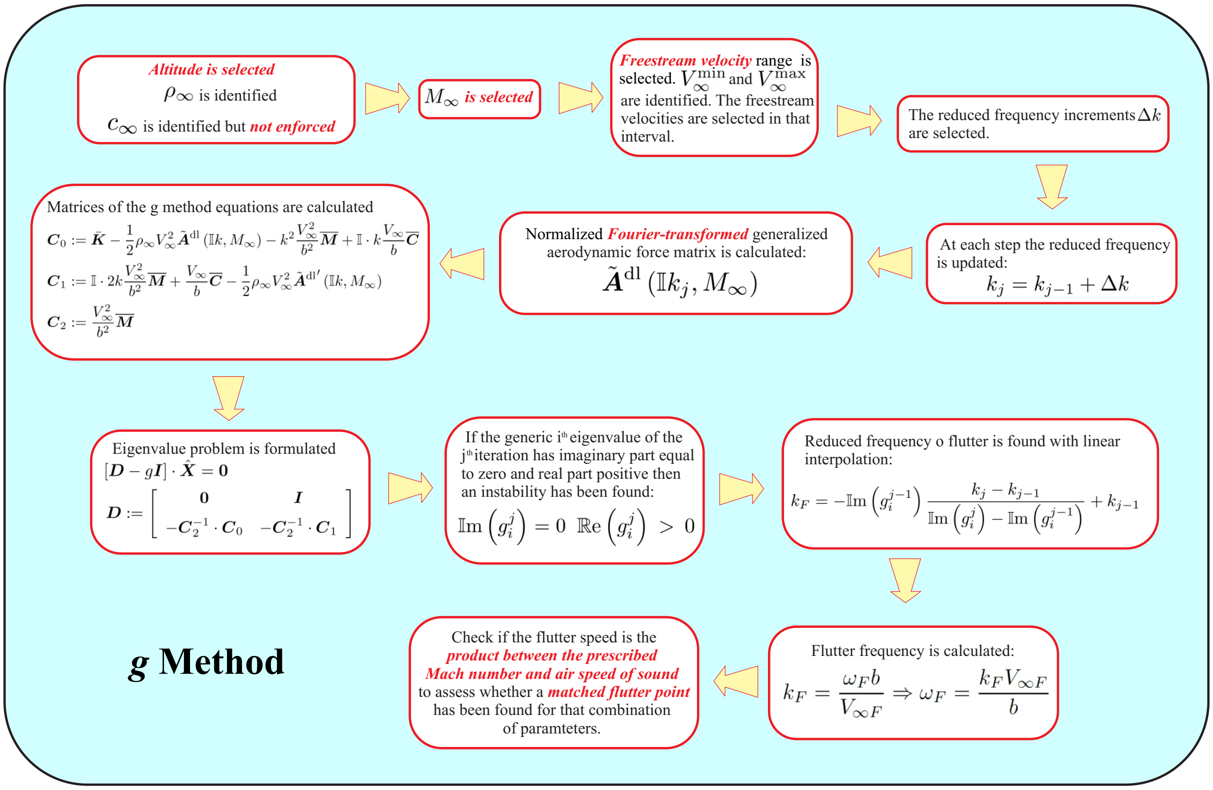

Figure 36.16: Summary of the conceptual steps of the the g method.

- *Step # 7*: If $\mathbb{I}\mathrm{m}\left(g_i^j\right) = 0$ (in practice this is satisfied within a tolerance) and $\mathbb{R}\mathrm{e}\left(g_i^j\right) > 0$ then the system is unstable and flutter occured. To find the correct reduced frequency of instability a linear interpolation is performed (similar to the one discussed in chapter 35 when the root locus was presented). More in detail, the linear interpolation is done as follows:

$$\frac{k_j - k_{j-1}}{\mathbb{I}\mathrm{m}\left(g_i^j\right) - \mathbb{I}\mathrm{m}\left(g_i^{j-1}\right)} = \frac{k_F - k_{j-1}}{0 - \mathbb{I}\mathrm{m}\left(g_i^{j-1}\right)} \tag{36.185}$$

where it was used the fact that at flutter $\mathbb{I}\mathrm{m}\left(g_F\right) = 0$. From equation 36.185 the flutter frequency is trivially obtained:

$$k_F = -\mathbb{I}\mathrm{m}\left(g_i^{j-1}\right)\frac{k_j - k_{j-1}}{\mathbb{I}\mathrm{m}\left(g_i^j\right) - \mathbb{I}\mathrm{m}\left(g_i^{j-1}\right)} + k_{j-1} \tag{36.186}$$

and it follows that the fixed speed used in the calculation is the flutter velocity $V_{\infty F} = V_\infty$. From the definition of reduced frequency the flutter frequency is also determined:

$$k_F = \frac{\omega_F b}{V_{\infty F}} \Rightarrow \omega_F = \frac{k_F V_{\infty F}}{b} \tag{36.187}$$

Notice that the flutter speed in general violates the condition $M_\infty = V_{\infty F}/c_\infty$. Thus several analyses need to be conducted to find the matched flutter point.

Figure 36.16 shows the main steps used in the g method.

36.240 (because is valid in general). However, the integrals H_0 and J_0 are determined as follows:

$$
\begin{aligned}
&J_0\left(u_1, k_1\right) = \frac{2}{k_1^2}\left(e^{\mathbb{I} k_1 u_1} - \mathbb{I} k_1 u_1 - 1\right) + \sum_{s=1}^{n} \frac{a_s\left(e^{\mathbb{I} k_1 u_1} + e^{c_s u_1}\left(u_1\left(c_s - \mathbb{I} k_1\right) - 1\right)\right)}{\left(c_s - \mathbb{I} k_1\right)^2} + e^{\mathbb{I} k_1 u_1} J_0\left(0, k_1\right) \\
&H_0\left(u_1, k_1\right) = \frac{2\mathbb{I}}{k_1}\left(e^{\mathbb{I} k_1 u_1} - 1\right) - \sum_{s=1}^{n} \frac{a_s\left(e^{\mathbb{I} k_1 u} - e^{-c_s u_1}\right)\left(c_s - \mathbb{I} k_1\right)}{c_s^2 + k_1^2} + e^{\mathbb{I} k_1 u_1} H_0\left(0, k_1\right) \\
&u_1 < 0
\end{aligned}
\tag{36.247}
$$

With the modified DLM code, now formulated directly in the Laplace domain, it is possible to generate root loci similar to the ones shown in Figure 35.8. The Generalized Aeroelastic Analysis Method has the advantage to provide an accurate estimate of the true aeroelastic damping, on the contrarty for example to the $p-k$ and g methods, which are less reliable for Lapace variables with significant positive or negative damping.

36.12 The p, $p-L$, and $p-p$ Methods

36.12.1 p Method

The set of aeroelastic equations written in the Laplace domain are the following (see equation 36.106):

$$\left[s^2\overline{\boldsymbol{M}} + s\overline{\boldsymbol{C}} + \overline{\boldsymbol{K}} - \frac{1}{2}\rho_\infty V_\infty^2 \hat{\boldsymbol{A}}^{\text{dl}}\left(\frac{sb}{V_\infty}, M_\infty\right)\right] \cdot \hat{\boldsymbol{q}} = \boldsymbol{0} \tag{36.248}$$

We have discussed that GAMM (see section 36.11) modifies the Doublet Lattice Method to directly obtain $\hat{\boldsymbol{A}}^{\text{dl}}\left(\frac{sb}{V_\infty}, M_\infty\right)$. Thus, equation 36.248 becomes a quadratic eigenvalue problem with complex matrix $\hat{\boldsymbol{A}}^{\text{dl}}\left(\frac{sb}{V_\infty}, M_\infty\right)$ and real matrices $\overline{\boldsymbol{M}}$, $\overline{\boldsymbol{C}}$, and $\overline{\boldsymbol{K}}$. Another approach, named p method uses constant matrices that multiply rational functions to build the normalized Laplace-transformed generalized aerodynamic force matrix $\hat{\boldsymbol{A}}^{\text{dl}}\left(\frac{sb}{V_\infty}, M_\infty\right)$:

$$\hat{\boldsymbol{A}}^{\text{dl}}\left(\frac{sb}{V_\infty}, M_\infty\right) = f_0\left(s\right) \cdot \boldsymbol{A}_0 + f_1\left(s\right) \cdot \boldsymbol{A}_1 + f_2\left(s\right) \cdot \boldsymbol{A}_2 + \ldots + f_N\left(s\right) \cdot \boldsymbol{A}_N \tag{36.249}$$

where the matrices $\boldsymbol{A}_i$ are constant and determined with an appropriate method. As an example, we have seen Roger's approximation and analytical continuation (see equation 30.22):

$$\hat{\boldsymbol{A}}^{\text{dl}}\left(s\right) = \boldsymbol{A}_0 + s\frac{b}{V_\infty}\boldsymbol{A}_1 + s^2\frac{b^2}{V_\infty^2}\boldsymbol{A}_2 + \sum_{m=1}^{N_{\text{lag}}} \frac{s}{s + \overline{\beta}_m}\boldsymbol{A}_{2+m} \tag{36.250}$$

By direct comparison between equations 36.249 and 36.250, it is realized that Roger's approximation[29] and the analytical continuation represent an application of the p method (see complete discussion of the root locus technique made in chapter 35). Notice that Roger's approximation is not the only approach that could be used to achieve the goal, see for example Karpel's Minimum State Method[30].

In some publications the p method is formulated in different forms[31]. Thus, the reader needs to be aware that in some publications different theoretical approaches are also classified to be the p method.

[29] Roger, K. L. "*Airplane Math Modeling Methods for Active Control Design*" AGARD Rept. 228, Neuilly sur-Seine, France, 1977.

[30] See Mordechay Karpel, "*Design for Active Flutter Suppression and Gust Alleviation Using State-Space Aeroelastic Modeling*", Journal of Aircraft, Vol. 19, No.3, 1982; Vepa R., "*Finite State Modeling of Aeroelastic Systems*", NASA CR-2779, 1977.

[31] H. Haddadpour, R. D. Firouz-Abadi, "*True damping and frequency prediction for aeroelastic systems: the $p-p$ method* ", Journal of Fluids and Structures, 25, 2009, pp. 1177-1188.

36.12.2 $p-p$ Method

The idea of the $p-p$ method[32] is to write the Laplace-transformed aeroelastic equations of motion (see equation 36.248) by introducing the dimensionless Laplace variable (which was defined in equation 36.155):

$$\left[\frac{V_\infty^2}{b^2}\left(s^{\mathrm{dl}}\right)^2\overline{\boldsymbol{M}}+\frac{V_\infty}{b}s^{\mathrm{dl}}\,\overline{\boldsymbol{C}}+\overline{\boldsymbol{K}}-\frac{1}{2}\rho_\infty V_\infty^2\hat{\boldsymbol{A}}^{\mathrm{dl}}\left(s^{\mathrm{dl}},M_\infty\right)\right]\cdot\hat{\boldsymbol{q}}=\mathbf{0} \tag{36.251}$$

The imaginary part of the normalized Laplace-transformed generalized aerodynamic force matrix is rewritten by considering the following identity (the arguments of the normalized Laplace-transformed generalized aerodynamic force matrix are omitted to have compact writings of the formulas):

$$\mathbb{I}\cdot\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)=\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{\frac{\omega b}{V_\infty}}\frac{\left(\sigma+\mathbb{I}\omega\right)b}{V_\infty}-\frac{\sigma}{\omega}\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)=\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{k}s^{\mathrm{dl}}-\frac{\sigma}{\omega}\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right) \tag{36.252}$$

Notice that when $k\to 0$ we can obtain the following limit[33] (see equations 36.155 and 36.166):

$$\lim_{k\to 0}\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{k}=\left.\frac{\partial\,\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{\partial k}\right|_{s^{\mathrm{dl}}=g} \tag{36.253}$$

Recall that by definition of complex numbers it is

$$\hat{\boldsymbol{A}}^{\mathrm{dl}}=\mathbb{R}\mathrm{e}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)+\mathbb{I}\cdot\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)=\mathbb{R}\mathrm{e}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)+\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{k}s^{\mathrm{dl}}-\frac{\sigma}{\omega}\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right) \tag{36.254}$$

where equation 36.252 was used.

Substituing equation 36.254 into equation 36.251:

$$\left[\frac{V_\infty^2}{b^2}\left(s^{\mathrm{dl}}\right)^2\overline{\boldsymbol{M}}+\frac{V_\infty}{b}s^{\mathrm{dl}}\,\overline{\boldsymbol{C}}+\overline{\boldsymbol{K}}-\frac{1}{2}\rho_\infty V_\infty^2\left(\mathbb{R}\mathrm{e}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)+\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{k}s^{\mathrm{dl}}-\frac{\sigma}{\omega}\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)\right)\right]\cdot\hat{\boldsymbol{q}}=\mathbf{0} \tag{36.255}$$

Rearranging the matrices:

$$\left[\frac{V_\infty^2}{b^2}\left(s^{\mathrm{dl}}\right)^2\overline{\boldsymbol{M}}+\frac{V_\infty}{b}\left(\overline{\boldsymbol{C}}-\frac{1}{2}\rho_\infty V_\infty b\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{k}\right)s^{\mathrm{dl}}+\left(\overline{\boldsymbol{K}}+\frac{1}{2}\rho_\infty V_\infty^2\left(-\mathbb{R}\mathrm{e}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)+\frac{\sigma}{\omega}\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)\right)\right)\right]\cdot\hat{\boldsymbol{q}}=\mathbf{0} \tag{36.256}$$

The following definitions are now introduced:

$$\breve{\boldsymbol{M}}:=\frac{V_\infty^2}{b^2}\overline{\boldsymbol{M}}\qquad\breve{\boldsymbol{C}}:=\frac{V_\infty}{b}\overline{\boldsymbol{C}}-\frac{1}{2}\rho_\infty V_\infty^2\frac{\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)}{k}\qquad\breve{\boldsymbol{K}}:=\overline{\boldsymbol{K}}+\frac{1}{2}\rho_\infty V_\infty^2\left(-\mathbb{R}\mathrm{e}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)+\frac{\sigma}{\omega}\mathbb{I}\mathrm{m}\left(\hat{\boldsymbol{A}}^{\mathrm{dl}}\right)\right) \tag{36.257}$$

Substituting equation 36.257 into equation 36.256:

$$\left[\breve{\boldsymbol{M}}\left(s^{\mathrm{dl}}\right)^2+\breve{\boldsymbol{C}}s^{\mathrm{dl}}+\breve{\boldsymbol{K}}\right]\cdot\hat{\boldsymbol{q}}=\mathbf{0} \tag{36.258}$$

Notice that the matrices $\breve{\boldsymbol{K}}$, $\breve{\boldsymbol{C}}$, and $\breve{\boldsymbol{M}}$ are real.

[32] H. Haddadpour, R. D. Firouz-Abadi, "*True damping and frequency prediction for aeroelastic systems: the $p-p$ method* ", Journal of Fluids and Structures, 25, pp. 1177-1188, 2009.

[33] D. S. Carter, "*L'Hospital's Rule for Complex-Valued Functions*", the American Mathematical Monthly, Vol. 65, No. 4, pp. 264-266, 1958.